Forcing NS_{ω_1} is ω_1-Dense From Large Cardinals Part II

A Journey Guided by the Stars

Andreas Lietz
TU Wien

February 18, 2024

CMU Core Model Seminar

Table of contents

Recap

Notation

Convention

Ideal means normal uniform ideal on ω_1 in this talk.

- If $\mathcal{I}$ is an ideal then $\mathbb{P}_{\mathcal{I}}$ is the associated forcing. It is

$$P(\omega_1)/\sim_{\mathcal{I}} - \{[\emptyset]_{\sim_{\mathcal{I}}}\}$$

  with the order induced by inclusion. Here, $A \sim_{\mathcal{I}} B$ iff $A \triangle B \in \mathcal{I}$.
- If G is $\mathbb{P}_{\mathcal{I}}$-generic over V then $U_G = \{A \mid [A]_{\sim_{\mathcal{I}}} \in G\}$ is a V-ultrafilter which induces the generic ultrapower

$$j_G \colon V \to \mathrm{Ult}(V, U_G).$$

Main Result

Definition

An ideal $\mathcal{I}$ is ω_1-dense if $\mathbb{P}_{\mathcal{I}}$ has a dense subsets of size ω_1.
That is there is $\langle S_i \mid i < \omega_1 \rangle$ a sequence of subsets of ω_1 so that for any $A \in \mathcal{I}^+$ there is $i < \omega_1$ with $S_i \backslash A \in \mathcal{I}$.

Theorem (L.)

If there is an inaccessible κ which is a limit of $<\kappa$-supercompact cardinals then there is a stationary set preserving forcing $\mathbb{P}$ with

$$V^{\mathbb{P}} \models \text{"}\mathrm{NS}_{\omega_1} \textit{ is } \omega_1\textit{-dense"}.$$

Prior to this, the only known models in which NS_{ω_1} is ω_1-dense were $\mathbb{Q}_{\max}$-extensions of $L(\mathbb{R})$ assuming $\mathrm{AD}^{L(\mathbb{R})}$ (or of other canonical models of AD^+).

The Ansatz

Some Heuristics

Consider the $\mathbb{Q}_{\max}$-version of $(*)$:

Definition

$\mathbb{Q}_{\max}$-$(*)$ holds if $L(\mathbb{R}) \models \mathrm{AD}$ and there is a filter $G \subseteq \mathbb{Q}_{\max}$ generic over $L(\mathbb{R})$ so that

$$(M_G, \mathcal{I}_G) = (H_{\omega_2}, \mathrm{NS}_{\omega_1}).$$

- Recall that $(M_G, \mathcal{I}_G, f_G)$ is the direct limit along G.
- We have $(M_G, \mathcal{I}) = (H_{\omega_2}, \mathrm{NS}_{\omega_1})^{L(\mathbb{R})[G]}$ and that NS_{ω_1} is ω_1-dense in $L(\mathbb{R})[G]$.
- Hence $\mathbb{Q}_{\max}$-$(*)$ implies "NS_{ω_1} is ω_1-dense".

By Asperó-Schindler, $\mathrm{MM}^{++} \Rightarrow (*)$. There should be some forcing axiom FA which solves

$$\frac{\mathrm{MM}^{++}}{(*)} = \frac{\mathrm{FA}}{\mathbb{Q}_{\max}\text{-}(*)}.$$

So FA implies "NS_{ω_1} is ω_1-dense".

Some Heuristics II

- Iterate small nice-ish forcings up to a supercompact κ via a RCS-iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$.
- Invoke an iteration theorem to argue that ω_1 (and suitable additional structure) is preserved along the iteration.
- Employ Baumgartner's argument to get the forcing axiom.

Here, have "NS_{ω_1} is ω_1-dense" in $V^{\mathbb{P}}$ as witnessed by a sequence $\vec{S} = \langle S_i \mid i < \omega_1 \rangle$ of stationary sets. $\mathbb{P}$ is κ-cc so that already $\vec{S} \in V^{\mathbb{P}_\alpha}$ for some $\alpha < \kappa$.

- Most likely, NS_{ω_1} is not ω_1-dense in $V^{\mathbb{P}_\alpha}$.
- But then $\mathbb{P}_{\alpha,\kappa}$ **must kill stationary sets** of $V^{\mathbb{P}_\alpha}$.
- Also $\mathbb{P}_{\alpha,\kappa}$ **must preserve the Π_1-properties of $\vec{S}$ that hold in $V^{\mathbb{P}}$.**

Iterating while Killing Stationary Sets

The First Obstacle

For a stationary $S \subseteq \omega_1$, let $\mathrm{CS}(S)$ denote the forcing that shoots a club through S.

- Let $\omega_1 = \bigcup_n S_n$ be a partition into stationary sets.
- Consider the iteration $\mathbb{P} = \langle \mathbb{P}_n, \dot{\mathbb{Q}}_m \mid n \leqslant \omega, m < \omega \rangle$ where

$$\Vdash_{\mathbb{P}_n} \dot{\mathbb{Q}}_n = \mathrm{CS}(\omega_1 - \check{S}_n)$$

  (choose your favorite support).
- In $V^{\mathbb{P}}$, ω_1^V is the countable union of non-stationary sets.
- So ω_1^V is collapsed.
- Problem: At each step, we go back to V to kill a set from there.
- **Solution: Only kill stationary sets that were just added in the last step!**

The Second Obstacle

This is Shelah's example of an iteration of SSP forcings collapsing ω_1.

- First force a function $g_0 \colon \omega_1 \to \omega_1$ above all canonical functions. Then force some g_1 above all canonical functions, but below g_0. Continue like this, get

  $$\text{canonical functions} < g_n < g_{n-1} < \cdots < g_1 < g_0 \quad \text{mod } \mathrm{NS}_{\omega_1}$$

  at stage *n*. These forcings preserve stationary sets, but not all are semiproper. In the limit ω_1 is collapsed (as there is no infinite decreasing sequence of such functions).
  Solution: Mostly use forcings with good "regularity properties".

The Iteration Theorem

These are the only two obstacles!

Theorem (L.)

Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ *be a RCS-iteration of* ω_1*-preserving forcings and assume that for all* $\alpha < \gamma$*:*

- $\Vdash_{\mathbb{P}_{\alpha+1}} \mathrm{SRP}$
- $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ *preserves stationary sets from* $\bigcup_{\beta<\alpha} V[\dot{G}_\beta]$"

Then $\mathbb{P}$ *preserves* ω_1*.*

This is a "cheapo iteration theorem", but good enough for our purposes.

The Correct Regularity Property

SRP hides the relevant regularity property. What is it?

For now consider an iteration $\mathbb{P} = \langle \mathbb{P}_n, \dot{\mathbb{Q}}_m \mid n \leqslant \omega, m < \omega \rangle$ iteration of length ω of ω_1-preserving forcings that do not kill "old stationary sets".

- Want to argue somehow that $\mathbb{P}$ preserves ω_1.
- So must find countable $X \prec H_\theta$ and p so that

$$p \Vdash \check{X} \sqsubseteq \check{X}[\dot{G}].$$

Let $X \prec H_\theta$ countable with $\mathbb{P} \in X$. Want to find $p_n \in \mathbb{P}_n$ so that $(p_n)_{n<\omega}$ is decreasing in $\mathbb{P}$ and

$$p_n \Vdash_{\mathbb{P}_n} \check{X} \sqsubseteq \check{X}[\dot{G}_n].$$

Suppose in step n of this argument, have

- Next forcing $\mathbb{Q} = \dot{\mathbb{Q}}_n^{G_n}$
- $S \subseteq \omega_1$ is stationary, $S \in X[G_n]$ but $\Vdash_{\mathbb{Q}} \check{S} \in \mathrm{NS}_{\omega_1}$ and
- $\delta^{X[G_n]} := X[G_n] \cap \omega_1 \in S$.

Then there is no way to continue! Must avoid this at all cost!

The Correct Regularity Property II

So need to start with X which avoids this problem, i.e. if $S \in X$ and $\mathbb{Q}_0$ kills S then $\delta^X \notin S$. This is easily possible!
Our regularity property should hand us some $p_0 \in \mathbb{Q}_0$ with

$$p_0 \Vdash_{\mathbb{Q}_0} \check{X} \sqsubseteq \check{X}[\dot{G}_1].$$

Even then, we might end up with the same problem at the next step $X[G_1]$! So p_0 must moreover avoid this situation for $X[G_1]$!

Definition
Say that a countable $Y \prec H_\theta$ respects an ideal $\mathcal{I}$ if $\delta^Y \notin S$ whenever $S \in \mathcal{I} \cap Y$.

In other words, need that $X[G_1]$ respects the ideal $\{S \subseteq \omega_1 \mid \mathbb{Q}_1 \text{ kills } S\}$.

Respectful Forcing

Definition

Suppose $\mathbb{Q}$ is ω_1-preserving forcing. $\mathbb{Q}$ is **respectful** if: Whenever

- $Y \prec H_\theta$ countable, $\mathbb{Q} \in Y$, $p \in \mathbb{Q} \cap Y$
- $\dot{I} \in Y$ is a $\mathbb{Q}$-name for an ideal on ω_1.

Then one of the following:

1. There is $q \leqslant p$ and q forces

$$Y \sqsubseteq Y[G] \wedge Y[G] \text{ respects } \dot{I}^G$$

2. Or: Y does **not** respect $\dot{I}^p := \{S \subseteq \omega_1 \mid p \Vdash \check{S} \in \dot{I}\}$.

This is a very strong regularity property! If $\mathbb{Q}$ is respectful and preserves stationary sets then $\mathbb{Q}$ is semiproper, but semiproper forcings need not be respectful.

How to use Respectfulness

Let's get back to our toy problem. Start with $X \prec H_\theta$ with $\mathbb{P} \in X$ so that X respects $\{S \subseteq \omega_1 \mid \mathbb{Q}_0 \text{ kills } S\}$.
Let $\dot{I}$ be the $\mathbb{Q}_0$-name for

$$\{S \subseteq \omega_1 \mid \dot{\mathbb{Q}}_1^{G_1} \text{ kills } S\}.$$

Since $\dot{\mathbb{Q}}_1^{G_1}$ does not kill old sets, X **trivially respects** $\dot{I}^{\mathbb{1}_{\mathbb{Q}_0}} \subseteq V$.
If $\mathbb{Q}_0$ is respectful then find p_0 so that

$$p_0 \Vdash_{\mathbb{Q}_0} \check{X} \sqsubseteq \check{X}[\dot{G}_1] \wedge \check{X}[\dot{G}_1] \text{ respects } \dot{I}^{\dot{G}_1}.$$

We are back in the same situation, only one step further. Can chain these arguments together!

Lemma

If $\mathbb{P}$ is a countable support iteration of respectful forcings which do not kill old stationary sets then $\mathbb{P}$ preserves ω_1.

The Role of SRP

Unfortunately, RCS iterations of respectful forcings need not be respectful. But we can simply nuke this problem!

Theorem (L.)

If SRP holds then every ω_1-preserving forcing is respectful.

Proof: Let $\mathbb{Q}$ be ω_1-preserving, $Y \prec H_\theta$, $q \in \mathbb{Q} \cap Y$, $\dot{I} \in Y$ as in definition. Have to show:

- Either there is $r \leqslant q$ forcing $Y \sqsubseteq Y[G]$ respects $\dot{I}^G$
- or Y does not respect $\dot{I}^q$.

Let $\mu = (2^{|Q|})^+ \in Y$ and
$\mathcal{S} = \{Z \prec H_\mu \mid \nexists r \leqslant q \text{ forcing "} Z \sqsubseteq Z[G] \text{ respects } \dot{I}^G\text{"}\} \in Y$.
By SRP, can find continuous increasing $\vec{Z} = \langle Z_\alpha \mid \alpha < \omega_1 \rangle \in Y$ s.t.:

- $\mathbb{Q}, q, \dot{I} \in Z_0$
- $Z_\alpha \prec H_\mu$
- Either $Z_\alpha \in \mathcal{S}$ or there is no $Z_\alpha \sqsubseteq Z$ with $Z \in \mathcal{S}$.

The Role of SRP II

Proof (Continued).
Let $G \subseteq \mathbb{Q}$ generic, $q \in G$. Let $S = \{\alpha < \omega_1 \mid Z_\alpha \in \mathcal{S}\}$.
Claim: $S \in I := \dot{I}^G$

Proof. Suppose otherwise, $S \in I^+$. $\langle Z_\alpha[G] \mid \alpha < \omega_1 \rangle$ is continuous increasing sequence of elementary substructures of $H_\mu^{V[G]}$. Find club $C \subseteq \omega_1$ with $\alpha = \delta^{Z_\alpha} = \delta^{Z_\alpha[G]}$. For any $\alpha \in S \cap C$, can find $T_\alpha \in I \cap Z_\alpha[G]$ with $\alpha = \delta^{Z_\alpha[G]} \in T_\alpha$. By normality of I, there is $S_0 \subseteq S \cap C$ in I^+ and T so that $T_\alpha = T$ for $\alpha \in S_0$. But then $S_0 \subseteq T$, contradicting $T \in I$.

$\square$

<u>Case 1:</u> $\delta^Y \in S$. As $S \in \dot{I}^q \cap Y$, Y does not respect $\dot{I}^q$.
<u>Case 2:</u> $\delta^Y \notin S$. As $Z_{\delta^Y} \sqsubseteq Y \cap H_\mu$, $Y \cap H_\mu \notin \mathcal{S}$. Thus there is $r \leqslant q$ forcing $Y \sqsubseteq Y[G]$ and $Y[G]$ respects $\dot{I}^G$. $\square$

In L, $\mathrm{Add}(\omega_1, 1)$ is *not* respectful.

Additional Structure To Preserve

Recall that we first force a candidate $\langle S_i \mid i < \omega_1 \rangle$ which might witness "NS_{ω_1} is ω_1-dense" in the future. This cannot be any random collection of ω_1-many stationary sets.

Lemma (Tennenbaum (?))

If $\mathbb{P}$ is a forcing of size ω_1 which collapses ω_1 then there is a dense embedding $\pi \colon \mathrm{Col}(\omega, \omega_1) \to \mathbb{P}$.

- $\Rightarrow$ Better: First force a candidate $\pi \colon \mathrm{Col}(\omega, \omega_1) \to \mathcal{P}(\omega_1) \backslash \mathrm{NS}_{\omega_1}$. In the end, want $\dot{[]}_{\mathrm{NS}_{\omega_1}} \circ \pi \colon \mathrm{Col}(\omega, \omega_1) \to \mathbb{P}_{\mathrm{NS}_{\omega_1}}$ a dense embedding.
- This suggests we should isolate properties of π, and then iterate forcing preserving these properties of π.

Definition (Woodin)

$\diamondsuit(\omega_1^{<\omega})$ holds if there is an embedding $\pi\colon \mathrm{Col}(\omega,\omega_1) \to \mathcal{P}(\omega_1)\setminus\mathrm{NS}_{\omega_1}$ so that $\forall p \in \mathrm{Col}(\omega,\omega_1)$ there are stationarily many countable $X \prec H_{\omega_2}$ with

$$p \in \{q \in \mathrm{Col}(\omega,\omega_1) \cap X \mid \omega_1 \cap X \in \pi(q)\} \text{ is a filter generic over } X.$$

Lemma

Suppose $[\cdot]_{\mathrm{NS}_{\omega_1}} \circ \pi\colon \mathrm{Col}(\omega,\omega_1) \to \mathbb{P}_{\mathrm{NS}_{\omega_1}}$ *is a dense embedding. Then* π *witnesses* $\diamondsuit(\omega_1^{<\omega})$.

Proof Sketch.

Let $p \in \mathrm{Col}(\omega,\omega_1)$, $X \prec H_{\omega_2}$ countable so that $\omega_1 \cap X =: \delta^X \in \pi(p)$.
Let $A \subseteq \mathrm{Col}(\omega,\omega_1)$, $A \in X$, be a maximal antichain.
$\Rightarrow \mathcal{A} := [\cdot]_{\mathrm{NS}_{\omega_1}} \circ \pi[A] \subseteq \mathbb{P}_{\mathrm{NS}_{\omega_1}}$ is a max. antichain, thus $\triangle\mathcal{A}$ contains a club $C \in X$, so $\delta^X \in C$. It follows that there is $q \in X \cap A$ with $\delta^X \in \pi(q)$. $\square$

More generally $\diamondsuit(\mathbb{B})$ and $\diamondsuit^+(\mathbb{B})$

Definition

Let $\mathbb{B} \subseteq \omega_1$ be a forcing. $\diamondsuit(\mathbb{B})$ holds if there is an embedding $\pi \colon \mathbb{B} \to \mathcal{P}(\omega_1)\setminus\mathrm{NS}_{\omega_1}$ so that $\forall p \in \mathbb{B}$ there are stationarily many countable $X \prec H_{\omega_2}$ with

$$p \in \{q \in \mathbb{B} \cap X \mid \omega_1 \cap X \in \pi(q)\} \text{ is a filter generic over } X.$$

We call such X π-slim.
The stronger $\diamondsuit^+(\mathbb{B})$ holds if there is π witnessing $\diamondsuit(\mathbb{B})$ so that every $X \prec H_\theta$ with $f, \mathbb{B} \in X$ is π-slim.

Lemma

If $\diamondsuit(\omega_1^{<\omega})$ holds then $\diamondsuit(\mathbb{B})$ holds for every forcing $\mathbb{B} \subseteq \omega_1$ (but not necessarily $\diamondsuit^+(\mathbb{B})$).

Parametrized Properness

Definition

Suppose π witnesses $\diamondsuit(\mathbb{B})$. A forcing $\mathbb{P}$ is π-**proper** if: Whenever

- $X \prec H_\theta$ countable and π-slim, $\mathbb{P} \in X$
- $p \in \mathbb{P} \cap X$

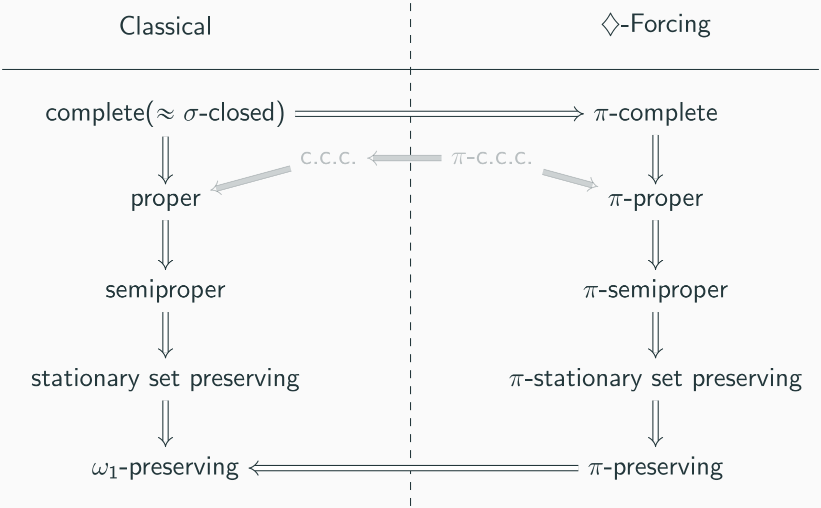
Then there is $(X, \mathbb{P}, \pi)$-generic $q \leqslant p$, i.e. forces

$$X = X[G] \cap V \wedge X[G] \text{ is } \pi\text{-slim.}$$

Analogously, define π-semiproperness.

Definition

Suppose π witnesses $\diamondsuit(\mathbb{B})$. A set $S \subseteq \omega_1$ is π-stationary if for large enough regular θ and all clubs $\mathcal{C} \subseteq [H_\theta]^\omega$ there is some π-slim $X \in \mathcal{C}$, $X \prec H_\theta$ with $\delta^X \in S$.


Classical
◊-Forcing
complete(≈ σ-closed)
π-complete
c.c.c.
π-c.c.c.
proper
π-proper
semiproper
π-semiproper
stationary set preserving
π-stationary set preserving
ω1-preserving
π-preserving

Iteration Theorems

Suppose π witnesses $\diamondsuit(\mathbb{B})$.

Theorem

Countable support iterations of π-proper forcings are π-proper

Theorem

RCS iterations of π-semiproper forcings are π-semiproper.

Thank you for listening!

To be continued...